Modular Arithmetic

Problem solving with Los A.V

In this tutorial we will solve a problem taken from a website[1]. The question is as follows:

> *We have a number of things, but we do not know exactly how many. If we count them by threes we have two left over. If we count by fives we have three left over. If we count by sevens there are two left over. How many things are there?*

This is a typical question where modular arithmetic (also Chinese Remainder Theorem) is involved. Please consider that you will need to have some basic knowledge about modular arithmetic.

By reading this problem, we can draw following congruent equations:

$$x \equiv 2\ (mod\ 3)$$

$$x \equiv 3\ (mod\ 5)$$

$$x \equiv 2\ (mod\ 7)$$

First of all, we know that in the first equation, $x = 3a + 2$, so we can insert that in the second equation instead of x.

$$3a + 2 \equiv 3\ (mod\ 5)$$

$$3a = 1\ (mod\ 5)$$

But now we have to calculate the value of a, by constructing a new equation.
$\{a \equiv b\ (mod\ n) \rightarrow n|a - b\}$. So:

$$5|3a - 1$$

What should a be equal to, in order to get a multiple of 5? Let's create a table.

a	$3a - 1$	
0	-1	X
1	2	X
2	5	V
3	8	X
4	11	X

As you might see, a should be 2. Please note, this will repeat itself, which means that a might be 7, 12, 17... as well. So we can write that as $a = 2 + 5z$. For example if z is 1, a is 7, which is a number in the sequence above.

Secondly, we should inset the value of a into $x = 3a + 2$, which will give us a new value of x.

$$x = 3(2 + 5z) + 2 = 6 + 15z + 2 = \mathbf{8 + 15z}$$

Thirdly, we inset the new value of x into the third equation which will give us:

[1] http://school.eb.co.uk/all/eb/article-9384387?query=modulo&ct=null

Artem Los, IB11C, Katedralskolan, Uppsala

$$8 + 15z \equiv 2\ (mod\ 7)$$

Please note that we need to subtract seven from left hand side.

$$1 + 1z = 2\ (mod\ 7)$$

$$z = 1\ (mod\ 7)$$

$$z = 1 + 7u$$

Now, we only need to plug in z into $x = 8 + 15z$, which gives us:

$$x = 8 + 15(1 + 7u) = 8 + 15 + 105u = 23 + 105u$$

So, the positive solutions for x are:

$$x = \{23{,}128, \ldots\}$$

So the answer is 23 things!

Artem Los, IB11C, Katedralskolan, Uppsala